A Framework to Understand the Retirement Age Debate*

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August 2010

Abstract

During the latest decades, most developed countries have experienced a fall in fertility and an increase in the longevity which have lead to a significant increase in the portion of elderly population and a decrease in the number of working-age people per elderly population. Economists and politicians are concerned about the future of the welfare state and the need to introduce policy reforms, such as fertility enhancing programs and delaying the retirement age. This paper determines the optimal retirement age and analyzes the effects of population aging on it. What is revealed is the different role that the drop in the fertility rate and the increase in the longevity play in the optimal retirement age. While an increase in the longevity implies an increase in the optimal retirement age, a drop in the fertility rate could imply a decrease in the optimal retirement age. We show that, in an infinite-horizon model populated with agents of different ages, if older workers are most productive and the average age of population is lesser than the average age of workers, the optimal response to a decrease in the fertility rate is a decrease in the retirement age. There are two mechanism behind: first, the increase in the portion of elderly has been offset by the decrease in the portion of the children, reducing the dependency ratio and second, a positive wealth effect thereby it increases the portion of older workers who are the most productive. This result shows the relevance of correctly analyzing the population and labor market structure in the design of the policies. And it shows that popular and widespread pronatalist policies among the developed countries are not always the solution of an aging population.

Keywords: Aging, Retirement age, Welfare Economics, Growth.
JEL Classification: E60, I30, J14, J26.

*We thank the participants in seminars at the Universidad de La Laguna for helpful comments.

1. Introduction

The fall in fertility experienced during the latest decades and the continuation of the rise in longevity has lead to a significant increase in the proportion of the older population in most of developed countries. Thus, the social security, the health care, long term care system and old age programs, whose expenditures are very much determined by the size of the older population, will come increasingly under financial stress. This is, if current fiscal rules are left unchanged while the portion of elderly people to workers continues growing, the public deficit and debt levels will continue to increase and will eventually explode. Policy reforms are thus necessary. This paper shows which is the optimal retirement age that economies must implement in order to reach the optimal size of the work force relative to the whole population. Moreover, it analyzes in detail which is the effect of the aging population on the optimal retirement age, this is, it separates the effect of a decrease in the fertility rate from the effect of an increase in the life expectancy.

On July 7th, 2010, newspapers around the world echoed the latest report from the European Commission which suggests the need to rise the average retirement age in the 27-nation bloc from the current age of 60 to 70 by 2060 if workers are to continue supporting retirees at current rates. At the moment there are four working-age people for every person over 65 in the 27-nation EU. That ratio will drop to two for every person over 65 by 2060. EU Employment Commissioner Laszlo Andor said it was urgent for Europe to act now because its working age population will start to shrink from 2012, he said: "The current situation is simply not sustainable". This concern about the welfare system sustainability is not a new issue [1] and something exclusive for Europe. The consequences of aging of baby-boomers in developed countries has been considered by politicians and economists for years [2]. However, the recent economic crisis has increased the doubts for the future of the welfare state and the need to introduce policy reforms.

Along the latest years, all of these countries have implemented a range of policies devoted to raise the fertility rate and thus, to slow down the aging of population and to increase the ratio of workers to retiree. Even the countries that could not be labeled of

[1] In latest years Germany, Netherlands and Denmark have deferred the retirement age until the 67, while United Kingdom has delayed the retirement age until 68. The government of Spain has considered to postpone the retirement age to 67 around 2040

[2] See for example, Velloso (2005), Weller (2005), Boldrin et al (2000), OECD (1998), European Comission Report (2004) and Congress of the United Status Report (2004).

pronatalist, like United States, have developed many family or social policies that may lead to higher fertility [3]. However, this paper shows that a higher fertility rate could not be the long run solution to reduce the retirement age. This is a very intriguing result which is at least surprising since it runs against the widespread public opinion. The reason is that while the ratio of retiree to workers decreases when the fertility increases, the ratio of children to workers increases as well. Therefore, the dependency ratio, defined as the number of people aged 0 to 14 and those aged 65 and over on the number of people aged 15 to 64, would result ambiguous. In the worst case the dependent population (those typically not in the work force) would increase more than the work force which could produce a higher pressure on the resources of the economy. [4]

This paper shows that if the average age of workers weighted by their contribution to the total labor supply is bigger than the average age of total population then, the optimal response to an increase in the fertility rate would be an increase in the retirement age. The intuition behind of this proposition is as follows. When the fertility rate rises, the weight that younger people has in population increases and the weight that elder people has in population goes down. This has three effects. First, an increase in the fertility rate will imply an increase in the child dependency rate. Second, an increase in the fertility rate will imply a reduction in the fraction of the older workers in the working population. Since older workers are more productive due to experience, the increase in fertility may reduce the productivity of labor The paper also establishes a precise condition under which an increase in the fertility rate reduces productivity. Finally, the increase of the fertility rate will reduce the elder dependency rate. Summarizing, when the fertility rate goes up, there are two mechanisms that reduce per capita labor: the increase in child dependence rate and the possible drop in the labor productivity. While there is a third mechanism that increases per capita labor when the fertility rate rises, higher fertility reduces the elder dependency rate. Thus, an important contribution of this paper is the establishment of a precise condition that may be easily obtained from the data (see table 1.1) and that determines the sign of the effect of the fertility rate over per capita labor.

[3]To this respect Grant, Hoorens, Sivadasan, Van Het Loo, Davanzo, Hale and Butz (2006) identifies serveral forms of policy interventions in familiy life or population structure. See Kohler, Billari and Ortega (2006) for the case of Europe, Suzuki (2004) for Japan and Fustos (2010) for United States.

[4]Fehr, Jokish and Kotlikoff (2008) simulates alternative fertility trends for developed countries in a social security OLG model. Results show that although a higher fertility rate increases the per capita work force, it increases expenditures for education, etc. which offsets the previous positive effect.

The establishment of this condition provides a precise criterion for political advice in pronatalist policies and greatly clarifies the debate over retirement age.

Assumptions behind this proposition are very reasonable and are supported by the existent empirical evidence. The first and the second columns in Table 1.1 show the average age of population and the average age of working population respectively for a wide sample of countries in 2005. The result is overwhelming: in almost all of countries the average age of working population is bigger than the average age of population. This aging of the work force has been well documented in the literature (see for example, Börsch-Supan, 2002 and Prskawetz, Fent and Guest, 2008).

Table 1.1: Average age population and average age of work force

Countries[a]	Av. age of population	Av. age of work force	Av. age of labor supply
Australia	37.07	39.60	41.76
Canada	38.65	40.41	42.70
Denmark	39.02	41.30	42.77
Finland	39.97	41.39	43.54
France	38.83	39.75	41.50
Germany	41.74	41.01	42.67
Italy	42.18	39.76	41.30
Japan	42.40	41.80	43.36
Netherlands	38.70	39.89	40.45
Spain	39.96	38.61	41.45
Sweden	40.53	42.03	44.04
United Kingdom	39.00	40.40	41.27
United States	36.40	40.29	42.88

[a]Source: Data on population weights are obtained in the US Census Bureau. Working population is measured as the active population aged from 20 to 64 years old. Data on activity rates are obtained in the OECD database. Relative weights of age workers groups used to obtain the average age of the labor supply have been calculated using the relative hourly wages by age groups. Data on the wage profiles are obtained from OECD report (1998).

Similarly, there exists abundant empirical evidence in favor of a hump-shaped wage-productivity profile in developed countries. The pattern implies that the wage-productivity for older workers is bigger than the average of the whole age profile [5]. Thus, the contribution of one old worker to the total labor supply in the economy is bigger than the average

[5]See for example Kydland (2004), Report of the National Equality Panel (2010), Blanchet et al. (2005), Luong and Hérbert (2009).

age worker. Third column in Table 1 shows the average age of the labor supply, this is, the average age of workers weighted by their productivity contribution. In all of cases (except Italy) the average age of the labor supply is bigger than the average age of work force. This result implies than contribution of the older workers to the labor supply is bigger that others and therefore, it makes older workers the most valuable ones for the economies.

We built a life infinite representative agent model in which agents decide consumption and their retirement age in each period. Individuals are born at age 0 and lives up to age $\overline{a}$. Since age 0 to age a_y they are children and do not work. Since age a_y to age a_o they are young active population and have a subjective cost for working which is increasing in age. However, they become retired from age a_r which is determined endogenously. From age a_o individuals face a survival probability which is decreasing in age up to $\overline{a}$. The first proposition of the paper is that an increase in the in the survival probability implies a raise in the retirement age, a_r. We also show that a technological improvement produces an increase in the retirement and the per capita capital. However, the most relevant result of the paper is that an increase in the fertility rate implies (for sufficient conditions) an increase in the dependency rate and as consequence, a raise in the retirement age, a_r.

Many papers in the literature have analyzed the relationship between retirement and social security. Earlier studies were focused on the effect of the introduction of a public pension program on the individual retirement decision (see for example Kahn 1988; Fabel 1994). Recent literature has studied the retirement decision from a political economy perspective (see Conde-Ruiz and Galasso 2003, 2004). Other papers have analyzed the retirement decision from a social security reform environment (see Auerbach, Kotlikoff, Hagemann, and G. Nicoletti, 1989 and De Nardi, Imrohoroglu, and Sargent, 1999). Finally, another recent strand of the literature has focused on the impact of changes in fertility and longevity in the definition of the retirement age. Lacomba and Lagos (2006) analyzes the effects of population aging on the retirement age using a life-cycle model. It finds that the demographic effects of a decrease in the population growth rate may lead to a delay in the preferred retirement age, when the dependence ratio modifies the contribution rate (see also Bloom, Canning and Graham, 2004 and Fehr, Jokish and Kotlikoff, 2008)

Few studies analyze the role of the intergenerational resources sharing to achieve intergenerational equity when population and productivity change. Theoretical papers provide no clear answers. Boadway, Marchand and Pestieau (1990 a, b) analyzes the ability of the social security to reallocated resources across generations when population and productivity vary over time. Meijdam and Verbon (1997) shows that the consequences of aging on the social security depends on the existing size of it. Marchand, Michel and Pestiau (1990) shows that the social security could make the equilibrium dynamically efficient. Crettez and Le Maitre (2002), also in a OLG model, shows that the retirement age chosen by a social planner is an increasing function of the population growth rate if the elasticity of substitution of old agents' labor for young agents' labor is lower than one. On the contrary, if the substitution of workers across age group is high and the size of a population does not matter, the optimal retirement age is a decreasing function of the population growth rate.

This lack of clear answers provided by the theoretical research is also reflected in the empirical results. Indeed, the emprical literature on the fiscal and economic effects of fertility changes is quite limited and yields controversial results. For example, Cutler, Poterba Sheiner and Summers (1990), Guest and McDonald (2002), Guest (2006) and Heijdra nd Ligthart (2006) find that declining fertility rates have a positive economic impact on future living standards, increasing the per capita consumption. While Berkel, Börsch-Supan, Ludwing and Winter (2004) finds that a drop in the fertility rate would worse the long-run pension finances.

The contribution of this paper is very important because is the first attempt to provide a theoretical analysis of the optimal retirement age determination in a general equilibrium model under uncertainty. Moreover, this study analyzes the effect of demographic changes on the optimal retirement age and how this effect may result in different conclusions depending on the population composition. The analysis on the effect of aging population is very meticulous distinguishing the increase in the fertility rate from an increase in the survival probabilities.

The paper is organized as follows. Section 2 develops a representative agent model. Section 3 derives the agents' decisions. Section 4 defines the equilibrium. Section 5 analyzes the dynamical behavior of the economy. Section 6 presents and discusses the results of an aging population. Section 7 analyzes the effects of an increase in the growth rate, and the last section concludes. The paper also includes a technical appendix.

2. The model

2.1. Demographic Dynamic:

Time is continuous and index by $t \in \Re$. Population is composed by agents of different ages, where $a \in [0, \overline{a}]$ is the support of ages; being $\overline{a}$ is the upper bound of duration of life. Live agents go through three stages: childhood, when $a \in [0, a_y)$; youth, when $a \in [a_y, a_o]$; and old age, when $a > a_o$, being $a_y < a_o$. Agents are able to work only during youth.

The probability of surviving at age a is denoted by $s(a)$. To simplify, we assume that agents during childhood and youth survive with probability one: $s(a) = 1, \forall a \leq a_o$; while during the old age the probability of being alive at age a is equal to $s(a) = \psi(a; \xi), \forall a > a_o$, where $\psi : [a_o, \overline{a}] \times \Re_+ \rightarrow [0, 1]$, is a function strictly decreasing in its first argument and strictly increasing in its second argument when $a \in (a_o, \overline{a})$ and such that $\psi(a_o; \xi) = 1$ and $\psi(\overline{a}; \xi) = 0$. Note that when ξ rises, the probability of being alive at age $a \in (a_o, \overline{a})$ also increases. Thus, an increase in ξ it implies an increase in the life expectancy. In this sense, parameter ξ could be considered like the health status or some biological ingredient that defines the individuals' capacity to survive. To simplify, we will interpret an increase in ξ as an increase in the life expectancy.

The amount of agents of age a at time t is denoted by $N(a, t)$ and is defined as follows:

$$N(a, t) = N(0, t - a)s(a) \tag{2.1}$$

That is, the amount of agents of age a at time t is equal to the amount of agent born a periods before, multiplied by the probability of being alive after a periods of being born.

We assume that births increase along time at a constat rate denoted by n:

$$\dot{N}(0, t) = nN(0, t) \tag{2.2}$$

Using (2.1) and (2.2) we get:

$$N(a, t) = s(a)N(0, t)e^{-na} \tag{2.3}$$

The total population $N(t)$ can be calculated as:

$$N(t) = \int_0^{\overline{a}} s(a)N(0, t)e^{-na} da$$

and, since 2.2 it is easy to see that the total population increases along time at the constant rate n:

$$\dot{N}(t) = nN(t) \tag{2.4}$$

Thus, the fraction of agents of age $\widehat{a}$ in the whole population, $\mu(\widehat{a})$, is defined by:

$$\mu(\widehat{a}) = \frac{s(\widehat{a})N(0,t)e^{-n\widehat{a}}}{\left[\int_0^{\overline{a}} s(a)N(0,t)e^{-na}da\right]} = \frac{s(\widehat{a})e^{-n\widehat{a}}}{\left[\int_0^{\overline{a}} s(a)e^{-na}da\right]}, \quad \forall \widehat{a} \leq \overline{a} \tag{2.5}$$

where:

$$\int_0^{\overline{a}} \mu(a)da = \int_0^{\overline{a}} \frac{s(a)e^{-na}}{\left[\int_0^{\overline{a}} s(a)e^{-na}da\right]}da = 1$$

We can rewrite the amount of agents of age a at time time t as a fraction of the total population at time t, this is,

$$N(a,t) = \mu(a)N(t) \tag{2.6}$$

2.2. Technology:

There is a unique good that may be used either as consumption good or as a investment good. There are two factors: labor L and capital K. The amount of production is given by the Cobb-Douglas production function:

$$\Gamma(t)^{1-\alpha}K(t)^{\alpha}L(t)^{1-\alpha}$$

There is exogenous technological change:

$$\dot{\Gamma}(t) = \gamma\Gamma(t)$$

The accumulation of capital follows the conventional neoclassical law of motion:

$$\dot{K}(t) = I(t) - \delta K(t)$$

where $I(t)$ denotes the gross investment and δ denotes the deprecation rate. If we define $k(t) \equiv \frac{K(t)}{N(t)}$ as the per capita capital, we may rewrite the capital accumulation equation as follows:

$$\dot{k}(t) = i(t) - (\delta + n)k(t)$$

where $i(t) \equiv \frac{I(t)}{N(t)}$ denotes the per capita investment. Taking into account that the production is devoted either to investment or to consumption, the above equation may be rewritten as follows:

$$\dot{k}\,(t) = \Gamma(t)^{1-\alpha} k(t)^{\alpha} l(t)^{1-\alpha} - c(t) - (\delta + n)k(t) \tag{2.7}$$

where $l(t) \equiv \frac{L(t)}{N(t)}$ and $c(t) \equiv \frac{C(t)}{N(t)}$ denote respectively per capita labor and per capita consumption.

2.3. Preferences and endowments:

Households maximize the sum of utility of their members, which depends on consumption:

$$\int_0^{\infty} \left[\int_{\Lambda(t)} N(a,t) \left[\ln(c(a,t)) - \phi(a)\right] da + \int_{[0,\overline{a}]-\Lambda(t)} N(a,t) \left[\ln(c(a,t))\right] da \right] e^{-\rho t} dt$$

where $c(a,t)$ denotes the consumption of the agents of age a at time t. $\Lambda(t)$ denotes the correspondence which relates the time index t with a measurable subset of $[a_y, a_o]$ which defines the ages at which agents work. Thus, if $a \in \Lambda(t)$, then agents of age a work at time t. In other words $\Lambda(t)$ defines the age of the work in the economy. Agents either work or not, but they can not chose the amount of time devoted to work which is exogenous. Finally, $\phi(a)$ is the desutility derived from working, which is an increasing function of age a. Furthermore, $\phi(.)$ is continuous and differentiable of second order, convex and $\lim_{a \to \overline{a}} \phi'(a) = +\infty$. This function could be interpreted as the subjective health cost of working. It is reasonable to assume that older workers have more health problems and thus, they have a lesser ability to work. [6]

Using the assumptions about the population behavior (see equations 2.4 and 2.6), we may rewrite the utility function as follows:

$$N(0) \left[\int_0^{\infty} \left[\int_0^{\overline{a}} \mu(a) \ln(c(a,t)) da - \int_{\Lambda_t} \mu(a)\phi(a) da \right] e^{-(\rho-n)t} dt \right]$$

Children and old age agents can not work. Young agents can work or not. We denote $l(a)$ the amount of labor that an young worker of age a owns, where $l(a)$ is a function

[6] There is many empirical evidence that establish a strong relationship between the ability to work and age (and health status). For example, Forman and Chen (2008) finds that more than half of men and one-third of women who left the labor force before the social security early-retirement age of 62 said that health limited their ability to work. Similarly, longitudinal data from the federal governments's Health and Retirement Survey shows that the onset of major health problems frequently leads directly to withdrawal from the labor force.

$l : [a_y, a_o] \rightarrow \Re_{++}$, which is continuous and differentiable of second degree and quasi-concave. We assume that $\frac{\phi(a)}{l(a)}$ is an increasing function. This assumption implies that the loss of working (the utility cost of working) is growing faster than the gain of working (the endowment of labor). Thus, besides the older workers are the most productive they suffer relatively more from health problems. At the end, the cost of working beats the gain of do it.

3. Agents Decisions

3.1. Households:

The household problem is as follows:

$$\max_{\{c(a,t)\}_{a=0}^{\overline{a}},\ \Lambda(t)} \int_0^{\infty} \left[\int_0^{\overline{a}} \mu(a) \ln(c(a,t))da - \int_{\Lambda(t)} \mu(a)\phi(a)da \right] e^{-(\rho-n)t}dt$$

$$\text{s.a.} \quad : \quad \dot{b}\ (t) = r(t)b(t) + w(t) \int_{\Lambda(t)} l(a)\mu(a)da - \int_0^{\overline{a}} \mu(a)c(a,t)da - nb(t) \qquad (3.1)$$

where $b(t)$ denotes the amount of assets per capita, $r(t)$ denotes the net interest rate, $w(t)$ denotes the wage per efficiency unit of labor and ρ denotes the subjective discount rate of the utility function, where $\rho > n$. The assumption that the instantaneous utility function derived from consumption $\ln(c)$ is strictly concave implies that consumption across agents will be identical at any optimal solution. The assumption that $\frac{\phi(a)}{l(a)}$ is an increasing function implies that at the optimal solution younger agents work, while older workers do not. These older agents that do not work are called retired. We can define the retirement age, $a_r(t)$, as the age such that agents work if they have an age equal or lower than $a_r(t)$ and they are retired in the opposite case, this is, older workers do not work while younger do it. As a consequence, the support of working ages $\Lambda(t)$ can be identified, this is, $\Lambda(t) = [a_y, a_r(t)]$.

All these facts allow us rewrite the household optimization problem as follows:

$$\max_{\{c(a,t)\}_{a=0}^{\overline{a}},\ a_r(t)} \int_0^{\infty} \left[\ln(c(t)) - \int_{a_y}^{a_r(t)} \mu(a)\phi(a)da \right] e^{-(\rho-n)t}dt \qquad (3.2)$$

$$\text{s.a.} \quad : \quad \dot{b}\ (t) = r(t)b(t) + w(t)l^s(a_r(t)) - c(t) - nb(t)$$

where $l^s(a_r(t)) = \int_{a_y}^{a_r(t)} l(a)\mu(a)da$ is the amount of per capita labor as a function of the retirement age, this is, the per capita labor supply in the economy [7]. Notice that the

[7]See Appendix A for a detailed description of the properties of labor supply function.

per capita labor supply is different from the per capita work force in the economy (active population), $z(a_r(t)) = \int_{a_y}^{a_r(t)} \mu(a)da$.[8]

The first order conditions and transversality conditions of the above problem are:

$$\begin{aligned} \frac{1}{c(t)} e^{-(\rho-n)t} &= \lambda(t) \\ \phi(a_r(t))e^{-(\rho-n)} &= \lambda(t)w(t)l(a_r(t)) \\ \dot{\lambda}\ (t) &= -\lambda(t)\left[r(t)-n\right] \\ \lim_{t\to+\infty} \lambda(t)b(t) &= 0 \end{aligned}$$

where $\lambda(t)$ are the Lagrangian multipliers of the associated Hamiltonian with the household optimization problem (3.2). Using the above first order conditions we get:

$$\phi(a_r(t)) = \frac{1}{c(t)} w(t)l(a_r(t)) \tag{3.3}$$

$$\frac{\dot{c(t)}}{c(t)} = [r(t)-\rho] \tag{3.4}$$

$$\left[\frac{\phi'(a_r(t))}{\phi(a_r(t))} - \frac{l'(a_r(t))}{l(a_r(t))}\right] \dot{a}_r\ (t) = -\left[r(t)-\rho\right] + \frac{\dot{w}\ (t)}{w(t)} \tag{3.5}$$

The first equation describes the trade off between the utility of not be working and the loss of labor income when an agent retires: if households decide that the retirement age is $a_r(t)$, the individuals who retire at that age increase their utility when they do not work in $-\phi(a_r(t))$. However, there is a lose in labor income equal to $w(t)l(a_r(t))$, which reduces consumption and therefore, the utility derived from consumption in $\frac{1}{c(t)}w(t)l(a_r(t))$. The second equation is the typical optimal rule of intertemporal consumption choice: the election between consumption at the present and consumption at the future depends on the degree at which consumption in the future is valuable , which depend on ρ, and the price at which consumption in the present may be substituted by consumption at the future. Finally, the third equation describes the trade off between the retirement at the present and the retirement at the future: if present generations retire later, this allow to accumulate assets and these assets allow future generation to retire earlier (this is why the return on assets r and the valuation of future generations ρ affects such choice). The second factor to consider is the growth rate of wages $\frac{\dot{w}(t)}{w(t)}$ which represents the difference

[8] If the endowments of labor along ages are equal to 1, $l(a) = 1, \forall a$, then the per capita labor supply coincides with the per capita work force.

in the lose of labor income that are going to suffer present and future generations when retire. If wages increase, future generation are going to earn more that present ones, and in this sense is better for present generation to retired earlier since the opportunity cost of retiring is not so high as in the future. The last factor to take into account is the trend in the net benefit resulting of combining the positive effect on the utility and the negative effect on the labor endowment (opportunity cost) when individuals retire at age $a_r(t)$, $\left[\frac{\phi'(a_r(t))}{\phi(a_r(t))} - \frac{l'(a_r(t))}{l(a_r(t))}\right]$. If the this net benefit is increasing at time t, individuals at present generation tends to retire later than individuals in future generations thereby the gains will be greater in the future. Note that one of the conditions (3.4) and (3.5) is redundant, since (3.3) and (3.4) imply (3.5); and (3.3) and (3.5) imply (3.4).

Finally, the transversality condition it may be rewritten in the following two alternative ways:

$$\begin{aligned}
\lim_{t\to+\infty} \frac{1}{c(t)} e^{-(\rho-n)t} b(t) &= 0 \\
\lim_{t\to+\infty} \frac{\phi(a_r(t))}{w(t)l(a_r(t))} e^{-(\rho-n)t} b(t) &= 0
\end{aligned}$$

and the interpretation is as usual: it means that nothing should be saved in the last period unless it is costless to do so (i.e., $\frac{1}{c(t)}e^{-(\rho-n)t} = \frac{\phi(a_r(t))}{w(t)l(a_r(t))}e^{-(\rho-n)t} = 0$).

3.2. Firms:

Firms are competitive and maximize profits:

$$\max_{K(t),\ L(t)} \Gamma(t)^{1-\alpha} K(t)^{\alpha} L(t)^{1-\alpha} - w(t)L(t) - (r(t)+\delta)K(t)$$

The first order condition of the above problem is:

$$\begin{aligned}
(1-\alpha)\Gamma(t)^{1-\alpha}\left(\frac{K(t)}{L(t)}\right)^{\alpha} &= w(t) \\
\alpha\left(\frac{\Gamma(t)L(t)}{K(t)}\right)^{1-\alpha} &= r(t)+\delta
\end{aligned}$$

The above equations are the well known conditions that say that firms should hire units of factor until the point in which the marginal productivity is equal to the price of use. These condition may be rewritten as follows:

$$(1-\alpha)\Gamma(t)^{1-\alpha}\left(\frac{k(t)}{l(t)}\right)^{\alpha} = w(t) \quad (3.6)$$

$$\alpha\left(\frac{\Gamma(t)l(t)}{k(t)}\right)^{1-\alpha} = r(t)+\delta \quad (3.7)$$

where $k(t)$ and $l(t)$ are respectively the per capita capital and labor.

4. Equilibrium

Definition: An equilibrium is a allocation $\{c(t), a_r(t), b(t), k(t), l(t), i(t)\}_{t=0}^{\infty}$ and a vector of prices $\{w(t), r(t)\}_{t=0}^{\infty}$ such that:

- Households maximize her utility: $\{c(t), a_r(t), b(t)\}_{t=0}^{\infty}$ is the solution of the household maximization problem (3.2)
- Firms maximize profits: $\{k(t), l(t)\}_{t=0}^{\infty}$ satisfy equations (3.6) and (3.7).
- Good market clears: $c(t) + i(t) = \Gamma(t)^{1-\alpha}k(t)^{\alpha}l(t)^{1-\alpha}$
- Labor market clears: $l^s(a_r(t)) = l(t)$
- Capital market clears: $b(t) = k(t)$

5. Dynamic Behavior

It follows from the equilibrium definition that the dynamic behavior of the economy may be describe by the following dynamic system:

$$\dot{a}_r\,(t) = \frac{(1-\alpha)(\delta+\rho+\gamma)+\alpha(\rho-n)-(1-\alpha)\alpha\left(\frac{\Gamma(t)l^s(a_r(t))}{k(t)}\right)^{1-\alpha}\frac{l(a_r(t))}{\phi(a_r(t))l^s(a_r(t))}}{\left[\frac{\phi'(a_r(t))}{\phi(a_r(t))}-\frac{l'(a_r(t))}{l(a_r(t))}+\alpha\frac{\mu(a_r(t))l(a_r(t))}{l^s(a_r(t))}\right]}$$

$$\dot{k}\,(t) = \Gamma(t)^{1-\alpha}k(t)^{\alpha}l^s(a_r(t))^{1-\alpha}\left[1-(1-\alpha)\frac{l(a_r(t))}{\phi(a_r(t))l^s(a_r(t))}\right]-(\delta+n)k(t)$$

$$\lim_{t\to+\infty}\frac{\phi(a_r(t))}{(1-\alpha)}\left(\frac{k(t)}{\Gamma(t)l^s(a_r(t))}\right)^{1-\alpha}e^{-(\rho-n)t} = 0$$

If we define $\widetilde{k}(t) \equiv \frac{k(t)}{\Gamma(t)}$, we may rewrite the system as follows:

$$\dot{a}_r\ (t) = \frac{(1-\alpha)(\delta+\rho+\gamma)+\alpha(\rho-n)-(1-\alpha)\alpha\left(\frac{l^s(a_r(t))}{\widetilde{k}(t)}\right)^{1-\alpha}\frac{l(a_r(t))}{\phi(a_r(t))l^s(a_r(t))}}{\left[\frac{\phi'(a_r(t))}{\phi(a_r(t))}-\frac{l'(a_r(t))}{l(a_r(t))}+\alpha\frac{\mu(a_r(t))l(a_r(t))}{l^s(a_r(t))}\right]} \quad (5.1)$$

$$\dot{\widetilde{k}}\ (t) = \widetilde{k}(t)^{\alpha}l^s(a_r(t))^{1-\alpha}\left[1-(1-\alpha)\frac{l(a_r(t))}{\phi(a_r(t))l^s(a_r(t))}\right]-(\delta+n+\gamma)\widetilde{k}(t) \quad (5.2)$$

$$\lim_{t\to+\infty}\frac{\phi(a_r(t))}{(1-\alpha)}\left(\frac{\widetilde{k}(t)}{l^s(a_r(t))}\right)^{1-\alpha}e^{-(\rho-n)t} = 0$$

Figure 5.1 describe the dynamic behavior of the system defined above. We see that the dynamic behavior is the typical saddle point dynamic. So, there is a unique equilibrium path that converge to the steady state, $(\widetilde{k}^*, a_r^*)$ [9]. We also can observe that along the equilibrium path the retired age decrease with the capital. This is due to the wealth effect: since early retirement is a normal good, when wealth increases agents retire earlier.

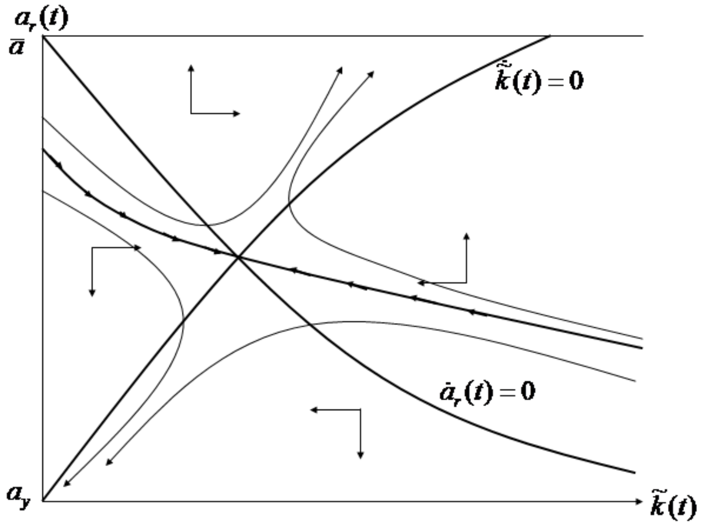

Figure 5.1: **Dynamic Behavior**

6. The effect of aging work force

In this section we study the effect of ageing of work force due to two different factors: an increase in the life expectancy and, a drop in the fertility rate. We show that these two

[9] See Appendix B for a detailed construction of the steady state of the economy.

sources have, under quite empirically plausible assumptions, exactly the opposite effects on the labor supply and the economy: an increase in the life expectancy increases the optimal retirement age at the steady state; while a drop in the fertility rate reduces the optimal retirement age at the steady state. An increase in life expectancy increase the portion of old agents and, therefore, given the retirement age, it increases the per capita retired agents, which implies an increase in the dependence rate. Thus, it has a negative "wealth effect": it reduces the per capita consumption level and it increases the retirement age at the steady state. The effect of a drop in the fertility rate on the economy is not clear. A drop in the fertility rate decreases the portion of younger people and thus, it increases the portion of retired population, while it decreases the portion of children in the population. However, the effect on the labor supply is ambiguous thereby it increases the portion of old workers and it decreases the portion of young workers. We prove that if the average age of labor supply is larger than the average age of total population then then, a drop in the fertility rate increases the labor supply in the economy. The reason is that a drop in the fertility rate gives more weight in the active population to older and more experienced and productive workers and thus, it produces an increase in labor productivity and in per capita labor.

6.1. The effect of an increase in the life expectancy:

Proposition 6.1. *At the steady state,* $\frac{\partial a_r^*}{\partial \xi} > 0$ *and* $\frac{\partial \widetilde{k}^*}{\partial \xi} < 0$.

Consider an increase in parameter ξ that implies an increase in the survival probabilities and therefore, an increase in the life expectancy. Notice that the increase in parameter ξ only affects the dynamic system (5.1)-(5.2) through the per capita labor supply. The labor supply is a decreasing function of ξ since an increase in life expectancy increases the old population and therefore, reduces the per capita work force[10]. Thus, an increase in life expectancy works as a negative wealth effect: it reduces the per capita consumption (and the per capita capital) and it increases the retirement age.

Figure 6.1 shows the effect of an increase of ξ over the steady state and the transitory dynamic. The increase of the life expectancy reduces the per capita labor supply, and this has a negative effect over the per capita resources of households, reducing consumption

[10]See Appendix A.

and increasing the retirement age.

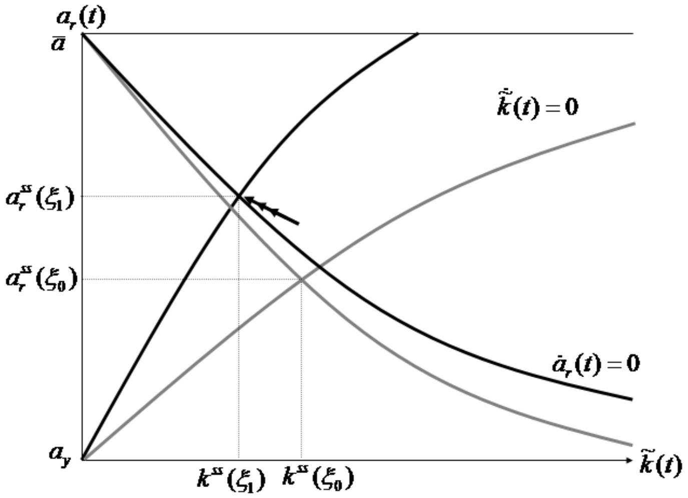

Figure 6.1: **The effect of an increase in life expectancy**

6.2. The effect of an increase in the fertility rate:

An increase in the fertility rate has two different mechanism: i) When the fertility rate increases, to keep the per capita capital constant requires a larger amount of investment. In this sense, the effect is similar to an increase in the depreciation rate. Thus, it is a negative "wealth effect". ii) The other mechanism is related with the effect of the fertility rate on the per capita labor supply. However, such effect is ambiguous. On the one hand, an increase in the fertility rate reduces the weight of old agents, increasing the portion of active population (work force) over the total population. On the other hand, it increases the weight of children, reducing the portion active population over total population. Thus, the effect on the dependency ratio would result ambiguous. Furthermore, because of the workers' labor productivity increases with experience (age), an increase in the fertility rate reduces the weight of more experienced and productive groups inside the labor market, reducing labor productivity an therefore, the per capita labor supply.

In order to analyze the effect of an increase in the fertility rate on the per capita labor supply, let's define the labor supply contribution density function as follows:

$$\nu(a;a_r) = \frac{l(a)\mu(a)}{\int_{a_y}^{a_r} l(a)\mu(a)da}$$

this density function gives the weight that each type-a agent has on the labor supply. Note that the more productive that an agent is, the higher her weight in the labor supply.

Proposition 6.2. *$\frac{\partial l^s(a_r)}{\partial n} < 0$ if and only if $E_\nu\left[a/\left[a_y,a_r\right]\right] > E\left[a\right]$, where $E_\nu\left[a/\left[a_y,a_r\right]\right] \equiv \int_{a_y}^{a_r} a\ \nu(a;a_r)\ da$ and $E\left[a\right] \equiv \int_0^{\overline{a}} a\ \mu(a)\ da$.*

Proposition 6.2 establishes that if the average age of the work force weighted by their contribution to the labor supply is larger than the average age of total population, then an increase on the fertility rate reduces the per capita labor supply. The above necessary condition implies the following sufficient one:

Corollary 6.3. *If $E\left[a/\left[a_y,a_r\right]\right] > E\left[a\right]$ and $l(a_r) > E\left[l(a)/\left[a_y,a_r\right]\right]$ then $\frac{\partial l^s(a_r)}{\partial n} < 0$.*

That is, if the average age of work force is larger than the average age of total population, and the amount of efficiency units of labor of agents at the retirement age is larger than the average amount of efficiency units of labor of work force, then an increase in fertility rate reduces the per capita labor supply. In this proposition we can see two different effects: first, when $E\left[a/\left[a_y,a_r\right]\right] > E\left[a\right]$, that is, when the average age of work force is larger than the average age of total population, an increase on the fertility rate reduces the proportion of the work force over the total population. Second, Furthermore, when $l(a_r) > E\left[l(a)/\left[a_y,a_r\right]\right]$, that is, when the number of efficiency units of labor of agents at retirement age is larger than the average on the age profile then, an increase on the fertility rate reduces the weight of the older and more experienced and productive workers which implies a drop in the average efficiency units of labor in the economy.

We notice that the necessary condition in proposition 6.2 is very reasonable. It is indeed supported by the existent empirical evidence. As we observed in Table 1.1, for all developed countries in the sample (except one), the average age of the work force weighted by their contribution to the labor supply is bigger than the average age of work force. Similarly, assumptions behind corollary 6.3 are also supported by the empirical evidence. Table 1.1 showed that in almost all of cases, the average age of working population is bigger than the average age of population. Moreover, the abundant empirical evidence in favor of a hump-shaped wage-productivity profile in developed countries supports the fact that the productivity at the retirement age is bigger than the average productivity along the life cycle. [11] Therefore, we can conclude that the most relevant case from the empirical point of view is the one in which an increase in fertility rate reduces the per capita labor supply. For this reason, we have centered the analysis in this case.

Proposition 6.4. *At the steady state, if* $\alpha < \frac{1}{2}$ *and* $\frac{\partial l^s(a_r)}{\partial n} < 0$ *then* $\frac{\partial a_r^*}{\partial n} > 0$ *and* $\frac{\partial \widetilde{k}^*}{\partial n} < 0$.

Considering the propositions above, a drop in the fertility rate has just the opposite effect that an increase in the life expectancy. If the fertility rate falls, the per capita amount of children falls as wells and therefore, it increases the per capita work force. Furthermore, the older and more productive workers gain weight at the work force, increasing the amount of efficient units of labor per worker, this is, increasing the per capita labor supply. Finally, the amount of investment needed to keep constant the amount of per capita capital is lesser. All these three effects represent positive wealth effects and therefore, produce a decrease in the optimal retirement age and an increase in the amount of per capita consumption and capital at the steady state. Figure 6.2 shows the new steady state and the dynamic transition due to a reduction in the fertility rate.

7. The effect of a technological improvement

Proposition 7.1. *At the steady state, if* $\alpha < \frac{1}{2}$ *then* $\frac{\partial a_r^*}{\partial \gamma} > 0$ *and* $\frac{\partial \widetilde{k}^*}{\partial \gamma} < 0$.

Imagine that growth rate of the technological change, γ increases. The effect on the capital per efficient unit of labor, $\widetilde{k}$ is clear: it produces a negative effect on the level of the capital per efficient unit of labor, equivalent to an increase in the depreciation rate, which implies a decrease in the capital per efficient unit of labor. However, the fact that the capital per efficient unit of labor remains constant at the steady state implies that the per capita capital is growing at the same rate than the technological progress. Thus, the rate or return at the steady state is higher and per capita consumption and wages are also increasing at the new higher rate.

[11] See Section 1 for a more detailed discussion about the suitability of the assumptions.

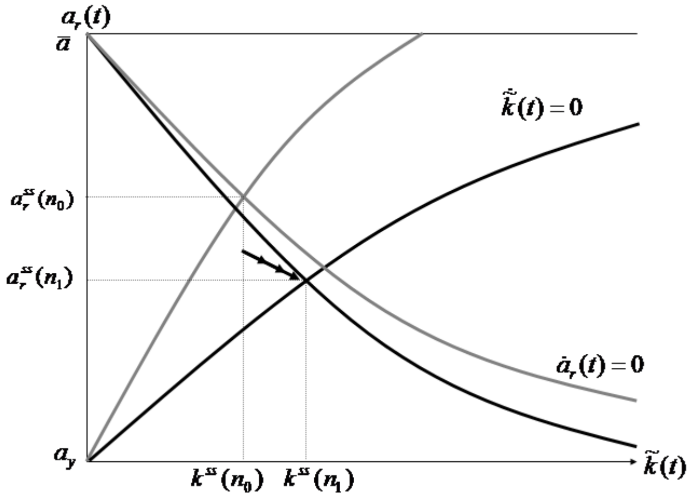

Figure 6.2: **The effect of a decrease in the fertility rate**

To study the effect on the retirement age we look at equation 3.5. Note that there is two effect running from different addresses: first, the increase in the return on assets implies an increase in the opportunity cost of retirement at the present. Agents would prefer to raise the retirement age at present and to decrease it in the future. Second, the increase in the wage growth rate makes more valuable to retire later in the future (with higher gains), so agents would be able to reduce the retirement age at present and to increase it in future. While the first effect prevails on the second at the beginning of the transition, after they offset each other and the retirement age remains stationary at the steady state. However, because of the rate of return in the steady state is higher, the retirement age increases as well.

8. Conclusion

The fall in fertility and the continuation of the rise in longevity experienced during the latest decades in developed countries has lead to a significant increase in the proportion of the older population and a decrease in the number of working-age people for every elder. This increase in the dependency ratio has arouse the debate of economists and politicians about the sustainabilility of the welfare state and the need to introduce policy reforms, such as the fertility enhancing programs and the delaying of the retirement age.

We have posed a growth model which determines the optimal retirement age that economies should implement in order to reach the optimal size of the work force relative to the whole population. Then, we have used the model to analyze the effect of the aging population on the optimal retirement age. We have considered two possible sources of aging: an increase in the life expectancy and a decrease in the fertility rate. We have made clear that these sources of ageing population have different effects. In fact, under quite empirically plausible assumptions, these two sources of ageing have exactly opposite effects. More precisely, an increase in the life expectancy increases the optimal retirement age at the steady state; while a drop in the fertility rate reduces the optimal retirement age at the steady state.

An increase in the life expectancy increases the portion of old agents which implies an increase the dependency rate, given the retirement age. This fact produces a negative "wealth effect" in the economy: it reduces per capita consumption and increases the retirement age at the steady state. The effect of a drop in the fertility rate is more complex. A drop in the fertility rate decreases the portion of younger people. Thus, it increases the portion of retired population and decreases the portion of children in the population. However, the effect on the portion of active population is ambiguous as well as on the dependency ratio. We show that if the life cycle wage profile is hump-shaped and the average age of the work force is larger than the average age of total population, a drop in the fertility rate increases the per capita labor in the economy. Furthermore, because of a drop in the fertility rate gives more weight to older and more experienced and productive workers in the work force, the drop in the fertility rate may produce an increase in labor productivity and in the per capita labor. Thus, the optimal solution for a drop in the fertility rate would be to increase the retirement age.

We have also analyzed an increase in the growth rate of the technical progress. It has a positive "wealth effect" implying an increase in the growth rate of per capita consumption and capital. Moreover, the higher rate of return in the steady state makes more valuable to work and retire later which implies an increase in the retirement age.

The findings of the paper show the relevance of correctly analyzing the population and labor market structure in the design of the policies. For example, the popular and widespread pronatalist policies among the developed countries could not be the solution for an increasing dependency ratio. In fact, under plausible assumptions, an increase in the fertility ratio could imply a negative "wealth effect" which would imply an increase in the retirement age. The empirical evidence we have added supports this result and shows that most developed countries are implementing wrong policies with opposite effects to the desired ones.

Appendix

A. The per capita labor supply

In this subsection we analyze the effect of different exogenous variables over the labor supply. The per capita labor supply is as follows:

$$l^s(a_r(t)) = \int_{a_y}^{a_r(t)} l(a)\mu(a)da = \frac{\int_{a_y}^{a_r(t)} l(a)s(a)e^{-na}da}{\left[\int_0^{\overline{a}} s(a)e^{-na}da\right]} = \frac{\int_{a_y}^{a_r(t)} l(a)e^{-na}da}{\left[\int_0^{a_o} e^{-na}da\right] + \left[\int_{a_o}^{\overline{a}} s(a,\xi)e^{-na}da\right]}$$

We can calculate the growth rate of labor supply as:

$$\frac{\dot{l^s}\ (a_r(t))}{l^s(a_r(t))} = \frac{l(a_r(t))\mu(a_r(t))\ \dot{a_r}\ (t)}{l^s(a_r(t))}$$

The effect of increasing life expectancy (increase ξ) is to reduce per capita labor supply:

$$\frac{\partial l^s}{\partial \xi} = \frac{-\left[\int_{a_y}^{a_r(t)} l(a)e^{-na}da\right]\left[\int_{a_o}^{\overline{a}} s'_\xi(a,\xi)e^{-na}da\right]}{\left[\left[\int_0^{a_o} e^{-na}da\right] + \left[\int_{a_o}^{\overline{a}} s(a,\xi)e^{-na}da\right]\right]^2} < 0$$

The effect of an increase in the fertility rate on the labor supply is not clear for two reasons. First, it reduces the portion of young individuals while increase the portion of old individuals, implying an ambiguous result on the work force, and; second, if the labor productivity is increasing with age ($l(a)$ is an increasing function) then, the increase of the fertility rate reduces the portion of highly productive agents while increase the portion of the less productive producing, again, an ambiguous result on the labor supply:

$$\frac{\partial l^s}{\partial n} = \frac{\left[\int_{a_y}^{a_r(t)} l(a)s(a)e^{-na}da\right]\left[\int_0^{\overline{a}} a\ s(a)e^{-na}da\right] - \left[\int_{a_y}^{a_r(t)} a\ l(a)s(a)e^{-na}da\right]\left[\int_0^{\overline{a}} s(a)e^{-na}da\right]}{\left[\int_0^{\overline{a}} s(a)e^{-na}da\right]^2} =$$

$$= \frac{\left[\int_{a_y}^{a_r(t)} l(a)s(a)e^{-na}da\right]}{\left[\int_0^{\overline{a}} s(a)e^{-na}da\right]}\left[\frac{\int_0^{\overline{a}} a\ s(a)e^{-na}da}{\int_0^{\overline{a}} s(a)e^{-na}da} - \frac{\int_{a_y}^{a_r(t)} a\ l(a)s(a)e^{-na}da}{\int_{a_y}^{a_r(t)} s(a)l(a)e^{-na}da}\right] =$$

$$= l^s(a_r(t))\left[\left(\int_0^{\overline{a}} a\ \frac{s(a)e^{-na}}{\left[\int_{a_y}^{a_r(t)} s(a)e^{-na}da\right]}da\right) - \left(\int_{a_y}^{a_r(t)} a\ \frac{l(a)s(a)e^{-na}}{\left[\int_{a_y}^{a_r(t)} l(a)s(a)e^{-na}da\right]}da\right)\right]$$

$$= l^s(a_r(t))\left(E\left[a\right] - E_\nu\left[a/\left[a_y, a_r(t)\right]\right]\right)$$

Thus the sign of the above derivative depends on the relationship between the average age of the population, $E[a]$, and the average age of the labor supply, $E_\nu[a/[a_y, a_r(t)]]$.

B. The steady state

This appendix describes the construction of the steady state. We first calculate and study the two stable loci ($\dot{a}_r = 0$ and $\dot{\widetilde{k}}_r = 0$) and then, we solve the resulting equations system. We prove that the solution is unique and so, that there is only one steady state.

B.1. Locus $\dot{a}_r = 0$

The locus along the retirement age remain constant is as follows:

$$\widetilde{k} = \left[\frac{(1-\alpha)\alpha}{[(1-\alpha)(\delta+\rho+\gamma)+\alpha(\rho-n)]}\frac{l(a_r(t))}{\phi(a_r)\,[l^s(a_r(t))]^\alpha}\right]^{\frac{1}{1-\alpha}}$$

The slope of the above locus is:

$$\frac{\partial\widetilde{k}}{\partial a_r} = \frac{1}{1-\alpha}\widetilde{k}\left[\frac{l'(a_r(t))}{l(a_r(t))} - \frac{\phi'(a_r(t))}{\phi(a_r(t))} - \alpha\frac{\mu(a)l(a_r(t))}{l^s(a_r(t))}\right] < 0$$

given that we have assumed that $\frac{\phi(a)}{l(a)}$ is an increasing function in a.

Furthermore, we can get the effect of different exogenous variables:

$$\frac{\partial\widetilde{k}}{\partial\xi} = -\frac{\alpha}{1-\alpha}\widetilde{k}\frac{\frac{\partial l^s(a_r(t))}{\partial\xi}}{l^s(a_r)} > 0$$

$$\frac{\partial\widetilde{k}}{\partial n} = \frac{\alpha}{1-\alpha}\widetilde{k}\left[-\frac{\frac{\partial l^s(a_r(t))}{\partial n}}{l^s(a_r)} + \frac{1}{[(1-\alpha)(\delta+\rho+\gamma)+\alpha(\rho-n)]}\right] \quad \text{ambiguous !}$$

$$\frac{\partial\widetilde{k}}{\partial\gamma} = \frac{\alpha}{1-\alpha}\widetilde{k}\left[\frac{1}{[(1-\alpha)(\delta+\rho+\gamma)+\alpha(\rho-n)]}\right] > 0$$

B.2. Locus $\dot{\widetilde{k}} = 0$:

$$\widetilde{k} = l^s(a_r(t))\left[\frac{\left[1-(1-\alpha)\frac{l(a_r(t))}{\phi(a_r(t))l^s(a_r(t))}\right]}{(\delta+n+\gamma)}\right]^{\frac{1}{1-\alpha}}$$

The slope of the above locus is:

$$\frac{\partial \widetilde{k}}{\partial a_r} = \widetilde{k}^{\alpha} \frac{\phi(a_r(t)) l(a_r(t)) \left[l^s(a_r(t))\right]^{\alpha}}{\delta + n + \gamma} \left[-\frac{l'(a_r(t))}{l(a_r(t))} + \frac{\phi'(a_r(t))}{\phi(a_r(t))} + \frac{\mu(a) l(a_r(t))}{l^s(a_r(t))} + \frac{\mu(a_r(t))}{\left[l^s(a_r(t))\right]^{2\alpha}} \right] > 0$$

Furthermore, we can get the effect of different exogenous variables:

$$\frac{\partial \widetilde{k}}{\partial \xi} = \widetilde{k}^{\alpha} \frac{1}{(\delta + n + \gamma) \left[l^s(a_r(t))\right]^{\alpha}} \left[1 + \alpha \frac{l(a_r(t))}{\phi(a_r(t)) l^s(a_r(t))} \right] \frac{\partial l^s(a_r(t))}{\partial \xi} < 0$$

$$\frac{\partial \widetilde{k}}{\partial n} = \widetilde{k}^{\alpha} \frac{1}{(\delta + n + \gamma) \left[l^s(a_r(t))\right]^{\alpha}} \left[1 + \alpha \frac{l(a_r(t))}{\phi(a_r(t)) l^s(a_r(t))} \right] \frac{\partial l^s(a_r(t))}{\partial n} - \frac{\widetilde{k}}{(1-\alpha)(\delta + n + \gamma)} \quad \text{ambiguous !}$$

$$\frac{\partial \widetilde{k}}{\partial \gamma} = -\frac{1}{1-\alpha} \frac{\widetilde{k}}{(\delta + n + \gamma)} < 0$$

B.3. Steady State

The steady state is the pair $(\widetilde{k}^*, a_r^*)$: $\dot{a}_r = 0$ and $\dot{\widetilde{k}}_r = 0$:

Retirement age:

$$\left[\frac{(1-\alpha)\alpha}{\left[(1-\alpha)(\delta + \rho + \gamma) + \alpha(\rho - n)\right]} \frac{l(a_r(t))}{\phi(a_r(t)) \left[l^s(a_r(t))\right]^{\alpha}} \right]^{\frac{1}{1-\alpha}} = l^s(a_r(t)) \left[\frac{\left[1 - (1-\alpha) \frac{l(a_r(t))}{\phi(a_r(t)) l^s(a_r(t))}\right]}{(\delta + n + \gamma)} \right]^{\frac{1}{1-\alpha}}$$

$$\begin{aligned} \frac{(1-\alpha)\alpha(\delta + n + \gamma)}{\left[(1-\alpha)(\delta + \rho + \gamma) + \alpha(\rho - n)\right]} \frac{l(a_r(t))}{\phi(a_r(t)) l^s(a_r(t))} &= \left[1 - (1-\alpha) \frac{l(a_r(t))}{\phi(a_r(t)) l^s(a_r(t))} \right] \\ \frac{\phi(a_r^*) l^s(a_r^*)}{l(a_r^*)} &= \frac{(1-\alpha)\left[\alpha(\delta + n + \gamma) + 1\right]}{\left[(1-\alpha)(\delta + \rho + \gamma) + \alpha(\rho - n)\right]} \end{aligned} \tag{B.1}$$

Capital:

$$\begin{aligned} \widetilde{k}^* &= l^s(a_r^*) \left[\frac{\left[1 - (1-2\alpha)(\delta + \rho + \gamma) - \alpha(\rho - n)\right]}{(\delta + n + \gamma)\left[\alpha(\delta + n + \gamma) + 1\right]} \right]^{\frac{1}{1-\alpha}} = \\ & \frac{(1-\alpha)\left[\alpha(\delta + n + \gamma) + 1\right]}{\left[(1-\alpha)(\delta + \rho + \gamma) + \alpha(\rho - n)\right]} \frac{l(a_r)}{\phi(a_r)} \left[\frac{\left[1 - (1-2\alpha)(\delta + \rho + \gamma) - \alpha(\rho - n)\right]}{(\delta + n + \gamma)\left[\alpha(\delta + n + \gamma) + 1\right]} \right]^{\frac{1}{1-\alpha}} \end{aligned} \tag{B.2}$$

C. Proof of proposition 6.1

- $\frac{\partial a_r^*}{\partial \xi} > 0$:

  From the definition of the loci $\dot{a}_r = 0$ and $\dot{\widetilde{k}} = 0$, in the steady state is verified that:

$$\widetilde{k}\Big|_{\dot{a}_r = 0} - \widetilde{k}\Big|_{\dot{\widetilde{k}} = 0} = 0$$

By the Implicit Function Theorem:

$$\frac{\partial a_r^*}{\partial \xi} = -\frac{\left.\frac{\partial \widetilde{k}}{\partial a_r}\right|_{\dot{a}_r=0} - \left.\frac{\partial \widetilde{k}}{\partial a_r}\right|_{\dot{\widetilde{k}}=0}}{\left.\frac{\partial \widetilde{k}}{\partial \xi}\right|_{\dot{a}_r=0} - \left.\frac{\partial \widetilde{k}}{\partial \xi}\right|_{\dot{\widetilde{k}}=0}} > 0$$

since $\left.\frac{\partial \widetilde{k}}{\partial a_r}\right|_{\dot{a}_r=0} < 0$, $\left.\frac{\partial \widetilde{k}}{\partial a_r}\right|_{\dot{\widetilde{k}}=0} > 0$, $\left.\frac{\partial \widetilde{k}}{\partial \xi}\right|_{\dot{a}_r=0} > 0$, $\left.\frac{\partial \widetilde{k}}{\partial \xi}\right|_{\dot{\widetilde{k}}=0} < 0$ (see the definition of the steady state in Appendix B for the calculations of the derivatives).

- $\frac{\partial \widetilde{k}^*}{\partial \xi} < 0$:

  From eq. B.2 it is easy to see that:

$$\frac{\partial \widetilde{k}^*}{\partial \xi} = \left[\frac{[1-(1-2\alpha)(\delta+\rho+\gamma)-\alpha(\rho-n)]}{(\delta+n+\gamma)[\alpha(\delta+n+\gamma)+1]}\right]^{\frac{1}{1-\alpha}} \frac{\partial l^s(a_r^*)}{\partial \xi}$$

then, using the definition of the retirement age at the steady state (eq. B.1) and the Implicit Function Theorem we get:

$$\frac{\partial l^s(a_r^*)}{\partial \xi} = \frac{(1-\alpha)[\alpha(\delta+n+\gamma)+1]}{[(1-\alpha)(\delta+\rho+\gamma)+\alpha(\rho-n)]} \frac{l(a_r^*)}{\phi(a_r^*)} \left[\frac{l'(a_r)}{l(a_r)} - \frac{\phi'(a_r)}{\phi(a_r)}\right] \frac{\partial a_r^*}{\partial \xi} < 0$$

since $\frac{\partial a_r^*}{\partial \xi} > 0$ as we proved above. Therefore, $\frac{\partial \widetilde{k}^*}{\partial \xi} < 0$.

D. Proof of proposition 6.2

From Appendix A we obtain that:

$$\begin{aligned}\frac{\partial l^s}{\partial n} &= l^s(a_r(t)) \left[\left(\int_0^{\overline{a}} a \frac{s(a)e^{-na}}{\left[\int_{a_y}^{a_r(t)} s(a)e^{-na}da\right]} da\right) - \left(\int_{a_y}^{a_r(t)} a \frac{l(a)s(a)e^{-na}}{\left[\int_{a_y}^{a_r(t)} l(a)s(a)e^{-na}da\right]} da\right)\right] = \\ &= l^s(a_r(t)) \left(E[a] - E_\nu [a/[a_y, a_r(t)]]\right)\end{aligned}$$

Therefore, if $E_\nu [a/[a_y, a_r(t)]] > E[a]$ then $\frac{\partial l^s}{\partial n} < 0$. ■

E. Proof of corollary 6.3

From Appendix A we obtain that:

$$\begin{aligned}\frac{\partial l^s}{\partial n} &= l^s(a_r(t)) \left(E[a] - E_\nu [a/[a_y, a_r(t)]]\right) = \\ &= l^s(a_r(t)) \left(E[a] - E[a/[a_y, a_r(t)]] + E[a/[a_y, a_r(t)]] - E_\nu [a/[a_y, a_r(t)]]\right)\end{aligned}$$

By assumption of the corollary $E[a/[a_y, a_r]] - E[a] > 0$, thus to prove the corollary only required to prove that $E_\nu [a/[a_y, a_r]] - E[a/[a_y, a_r]] \geq 0$.

$$E_\nu\left[a/\left[a_y,a_r\right]\right]-E\left[a/\left[a_y,a_r\right]\right]=\frac{\int_{a_y}^{a_r(t)} a\ l(a)\mu(a)da}{\int_{a_y}^{a_r(t)} l(a)\mu(a)da}-\frac{\int_{a_y}^{a_r(t)} a\ \mu(a)da}{\int_{a_y}^{a_r(t)} \mu(a)da}=$$

$$=\frac{\int_{a_y}^{a_r(t)} a\ l(a)\mu(a)da-\frac{\int_{a_y}^{a_r(t)} l(a)\mu(a)da}{\int_{a_y}^{a_r(t)} \mu(a)da}\int_{a_y}^{a_r(t)} a\ \mu(a)da}{\int_{a_y}^{a_r(t)} l(a)\mu(a)da}= \tag{E.1}$$

$$=\frac{\int_{a_y}^{a_r(t)} a\ l(a)\mu(a)da-E\left[l(a)/\left[a_y,a_r\right]\right]\int_{a_y}^{a_r(t)} a\ \mu(a)da}{\int_{a_y}^{a_r(t)} l(a)\mu(a)da}=$$

$$=\frac{1}{\int_{a_y}^{a_r(t)} l(a)\mu(a)da}\left[\int_{a_y}^{a_r(t)} a\ \left(l(a)-E\left[l(a)/\left[a_y,a_r\right]\right]\right)\mu(a)da\right] \tag{E.2}$$

It follows from Weierstrass Theorem and continuity of $l(a)$ that $l(a)$ reach a minimum in the interval $[a_y, a_r(t)]$. Since the function $l(a)$ is quasi-concave, the function reach a minimum either at a_y or at a_r:

$$l(a)=l\left(\lambda(a)l(a_y)+(1-\lambda(a))l(a_r)\right)\geq \min\left\{l(a_y),l(a_r)\right\}\quad \forall a\in(a_y,a_r)$$

where $\lambda(a)=\frac{a_r-a}{a_r-a_y}$. Since by assumption $l(a_r)>E\left[l(a)/\left[a_y,a_r\right]\right]$, the minimum is reached at a_y. Thus, $a_y<E\left[l(a)/\left[a_y,a_r\right]\right]$. Since $l(.)$ is continuous, there is $\widetilde{a}\in(a_y,a_r)$ such that $l(\widetilde{a})=E\left[l(a)/\left[a_y,a_r\right]\right]$ and $l(a)<E\left[l(a)/\left[a_y,a_r\right]\right]$ $\forall a\in[a_y,\widetilde{a})$. Furthermore it follows from quasi-concavity that:

$$l(a)=l\left(\widetilde{\lambda}(a)l(\widetilde{a})+(1-\widetilde{\lambda}(a))l(a_r)\right)\geq \min\left\{l(\widetilde{a}),l(a_r)\right\}=E\left[l(a)/\left[a_y,a_r\right]\right]\quad \forall a\in(\widetilde{a},a_r] \tag{E.3}$$

where $\widetilde{\lambda}(a)=\frac{a_r-a}{a_r-\widetilde{a}}$.

Using equation (E.2) yields:

$$\int_{a_y}^{a_r(t)} a\ \left(l(a)-E\left[l(a)/\left[a_y,a_r\right]\right]\right)\mu(a)da=$$

$$\int_{a_y}^{a_r(t)} a\ \left(l(a)-E\left[l(a)/\left[a_y,a_r\right]\right]\right)\mu(a)da-\widetilde{a}\int_{a_y}^{a_r(t)}\mu(a)da\underbrace{\left[\frac{\int_{a_y}^{a_r(t)} l(a)\mu(a)da}{\int_{a_y}^{a_r(t)} \mu(a)da}-E\left[l(a)/\left[a_y,a_r\right]\right]\right]}_{0}=$$

$$\int_{a_y}^{a_r(t)} a\ \left(l(a)-E\left[l(a)/\left[a_y,a_r\right]\right]\right)\mu(a)da-\widetilde{a}\left[\int_{a_y}^{a_r(t)} l(a)\mu(a)da-E\left[l(a)/\left[a_y,a_r\right]\right]\int_{a_y}^{a_r(t)}\mu(a)da\right]=$$

$$\int_{a_y}^{a_r(t)} a \;\left(l(a) - E\left[l(a)/\left[a_y, a_r\right]\right]\right)\mu(a)da - \widetilde{a}\left[\int_{a_y}^{a_r(t)} \left(l(a) - E\left[l(a)/\left[a_y, a_r\right]\right]\right)\mu(a)da\right]$$

$$\int_{a_y}^{a_r(t)} (a-\widetilde{a}) \;\left(l(a) - E\left[l(a)/\left[a_y, a_r\right]\right]\right)\mu(a)da \;=$$

$$\int_{a_y}^{\widetilde{a}} \underbrace{(a-\widetilde{a})}_{\ominus}\underbrace{\left(l(a) - E\left[l(a)/\left[a_y, a_r\right]\right]\right)}_{\ominus}\mu(a)da + \int_{\widetilde{a}}^{a_r(t)} \underbrace{(a-\widetilde{a})}_{\oplus}\underbrace{\left(l(a) - E\left[l(a)/\left[a_y, a_r\right]\right]\right)}_{\oplus}\mu(a)da \;>\; 0$$

where in the first equality we use the definition of $E\left[l(a)/\left[a_y, a_r\right]\right]$, and in the last equality we used the definition of $\widetilde{a}$ and equation (E.3).

F. Proof of proposition 6.4

- $\frac{\partial a_r^*}{\partial n} > 0$:

From the definition of the loci $\dot{a}_r = 0$ and $\dot{\widetilde{k}} = 0$, in the steady state is verified that:

$$\widetilde{k}\Big|_{\dot{a}_r=0} - \widetilde{k}\Big|_{\dot{\widetilde{k}}=0} = 0$$

By the Implicit Function Theorem:

$$\frac{\partial a_r^*}{\partial n} = -\frac{\frac{\partial \widetilde{k}}{\partial a_r}\Big|_{\dot{a}_r=0} - \frac{\partial \widetilde{k}}{\partial a_r}\Big|_{\dot{\widetilde{k}}=0}}{\frac{\partial \widetilde{k}}{\partial n}\Big|_{\dot{a}_r=0} - \frac{\partial \widetilde{k}}{\partial n}\Big|_{\dot{\widetilde{k}}=0}} > 0$$

since $\frac{\partial \widetilde{k}}{\partial a_r}\Big|_{\dot{a}_r=0} < 0$, $-\frac{\partial \widetilde{k}}{\partial a_r}\Big|_{\dot{\widetilde{k}}=0} > 0$, $\frac{\partial \widetilde{k}}{\partial n}\Big|_{\dot{a}_r=0} > 0$ thereby $\frac{\partial l^s(a_r^*)}{\partial n} < 0$ and finally, $\frac{\partial \widetilde{k}}{\partial n}\Big|_{\dot{\widetilde{k}}=0} < 0$ thereby $\frac{\partial l^s(a_r^*)}{\partial n} < 0$ (see the definition of the steady state in Appendix B for the calculations of the derivatives).

- $\frac{\partial \widetilde{k}^*}{\partial n} < 0$:

From Figure 1 it is easy to see that

$$\text{if} \quad \left|\frac{\partial \widetilde{k}}{\partial n}(\widetilde{k}^*)\Big|_{\dot{\widetilde{k}}=0}\right| > \frac{\partial \widetilde{k}}{\partial n}(\widetilde{k}^*)\Big|_{\dot{a}_r=0} \quad \text{then} \quad \frac{\partial \widetilde{k}^*}{\partial n} < 0$$

We now proceed to prove that $\left|\frac{\partial \widetilde{k}}{\partial n}(\widetilde{k}^*)\Big|_{\dot{\widetilde{k}}=0}\right| > \frac{\partial \widetilde{k}}{\partial n}(\widetilde{k}^*)\Big|_{\dot{a}_r=0}$. Since $\frac{\partial l^s(a_r)}{\partial n} < 0$ we can rewrite it as

$$\left|\widetilde{k}^{\alpha}\frac{1}{(\delta+n+\gamma)\left[l^s(a_r(t))\right]^{\alpha}}\left[1+\alpha\frac{l(a_r(t))}{\phi(a_r(t))l^s(a_r(t))}\right]\frac{\partial l^s(a_r(t))}{\partial n}\right| + \frac{\widetilde{k}}{(1-\alpha)(\delta+n+\gamma)} >$$

$$\frac{\alpha}{1-\alpha}\widetilde{k}\left[-\frac{\frac{\partial l^s(a_r(t))}{\partial n}}{l^s(a_r)} + \frac{1}{\left[(1-\alpha)\left(\delta+\rho+\gamma\right)+\alpha(\rho-n)\right]}\right]$$

$$\underbrace{\left|\widetilde{k}^{\alpha}\frac{1}{(\delta+n+\gamma)\left[l^s(a_r(t))\right]^{\alpha}}\left[1+\alpha\frac{l(a_r(t))}{\phi(a_r(t))l^s(a_r(t))}\right]\frac{\partial l^s(a_r(t))}{\partial n}\right|-\frac{\alpha}{1-\alpha}\widetilde{k}\frac{\frac{\partial l^s(a_r(t))}{\partial n}}{l^s(a_r)}}_{A}+$$

$$\underbrace{\frac{\widetilde{k}}{(1-\alpha)(\delta+n+\gamma)}-\frac{\alpha}{1-\alpha}\frac{\widetilde{k}}{\left[(1-\alpha)\left(\delta+\rho+\gamma\right)+\alpha(\rho-n)\right]}}_{B}>0$$

The strategy we follow is to prove that $A>0$ and $B>0$. We start with $B>0$:

$$\frac{\widetilde{k}}{(1-\alpha)(\delta+n+\gamma)}-\frac{\alpha}{1-\alpha}\frac{\widetilde{k}}{\left[(1-\alpha)\left(\delta+\rho+\gamma\right)+\alpha(\rho-n)\right]}>0$$

Then,

$$\delta+\rho+\gamma>2\alpha(\delta+n+\gamma)$$

and this is true since $\rho>n$ and $\alpha<\frac{1}{2}$.

Now we prove that $A>0$:

$$\left|\widetilde{k}^{\alpha}\frac{1}{(\delta+n+\gamma)\left[l^s(a_r(t))\right]^{\alpha}}\left[1+\alpha\frac{l(a_r(t))}{\phi(a_r(t))l^s(a_r(t))}\right]\frac{\partial l^s(a_r(t))}{\partial n}\right|-\frac{\alpha}{1-\alpha}\widetilde{k}\frac{\frac{\partial l^s(a_r(t))}{\partial n}}{l^s(a_r)}>0$$

$$\frac{1-\alpha}{\alpha(\delta+n+\gamma)}\left[1+\alpha\frac{l(a_r(t))}{\phi(a_r(t))l^s(a_r(t))}\right]>\frac{\widetilde{k}^{1-\alpha}}{\left[l^s(a_r(t))\right]^{1-\alpha}}$$

$$l^s(a_r(t))\left(\frac{1-\alpha}{\alpha(\delta+n+\gamma)}\left[1+\alpha\frac{l(a_r(t))}{\phi(a_r(t))l^s(a_r(t))}\right]\right)^{\frac{1}{1-\alpha}}>\widetilde{k}$$

Using the definition of capital at the steady state (eq. B.2)

$$l^s(a_r(t))\left(\frac{1-\alpha}{\alpha(\delta+n+\gamma)}\left[1+\alpha\frac{l(a_r(t))}{\phi(a_r(t))l^s(a_r(t))}\right]\right)^{\frac{1}{1-\alpha}}>l^s(a_r)\left[\frac{\left[1-(1-2\alpha)\left(\delta+\rho+\gamma\right)-\alpha(\rho-n)\right]}{(\delta+n+\gamma)\left[\alpha(\delta+n+\gamma)+1\right]}\right]^{\frac{1}{1-\alpha}}$$

$$\frac{1-\alpha}{\alpha(\delta+n+\gamma)}\left[1+\alpha\frac{l(a_r(t))}{\phi(a_r(t))l^s(a_r(t))}\right]>\frac{\left[1-(1-2\alpha)\left(\delta+\rho+\gamma\right)-\alpha(\rho-n)\right]}{(\delta+n+\gamma)\left[\alpha(\delta+n+\gamma)+1\right]}$$

Using the expression B.1

$$\frac{1-\alpha}{\alpha}\left[1+\alpha\frac{\left[(1-\alpha)\left(\delta+\rho+\gamma\right)+\alpha(\rho-n)\right]}{(1-\alpha)\left[\alpha(\delta+n+\gamma)+1\right]}\right]>\frac{\left[1-(1-2\alpha)\left(\delta+\rho+\gamma\right)-\alpha(\rho-n)\right]}{\left[\alpha(\delta+n+\gamma)+1\right]}$$

$$\frac{1-\alpha}{\alpha}+2\left[(1-\alpha)(\delta+\rho+\gamma)+\alpha(\rho-n)\right]>1-(1-2\alpha)(\delta+\rho+\gamma)$$

$$\frac{1-2\alpha}{\alpha}+2\left[(1-\alpha)(\delta+\rho+\gamma)+\alpha(\rho-n)\right]>-(1-2\alpha)(\delta+\rho+\gamma)$$

and this is true for $\alpha<\frac{1}{2}$.

- $\frac{\partial l^s(a_r^*)}{\partial n} < 0$:

In subsection 2.2 the per capita income is defined as $y(t) = \Gamma(t)^{1-\alpha} k(t)^{\alpha} l(t)^{1-\alpha}$. Thus, the per capita income growth rate is: $\frac{\dot{y}(t)}{y(t)} = (1-\alpha)\frac{\dot{\Gamma}(t)}{\Gamma(t)} + \alpha\frac{\dot{k}(t)}{k(t)} + (1-\alpha)\frac{\dot{l}(t)}{l(t)}$. If we evaluate it in the steady state we get: $\frac{\dot{y}(t)}{y(t)} = \gamma$, since $\dot{\widetilde{k}}= 0$ which implies $\frac{\dot{k}(t)}{k(t)} = \gamma$ and, $\dot{a}_r= 0$ which implies $\frac{\dot{l}(t)}{l(t)} = 0$ (see Appendix A).

From eq. 2.7 we get $c(t) = y(t)- \dot{k}\ (t) - (\delta + n)k(t)$. If we evaluate the per capita consumption at the steady state we get $\frac{c(t)}{k(t)} = \frac{y(t)}{k(t)} - (\gamma + \delta + n)k(t)$. It is easy to see that $\frac{\dot{c}(t)}{c(t)} = \frac{\dot{k}(t)}{k(t)} = \gamma$, since $\frac{\dot{y}(t)}{y(t)} = \frac{\dot{k}(t)}{k(t)} = \gamma$ which implies that the rate $\frac{y(t)}{k(t)}$ is constant and, thus the rate $\frac{c(t)}{k(t)}$ is constant as well.

Therefore, eq. 3.3 can be rewritten as: $r(t) = \gamma + \rho$. If we substitute $r(t)$ defined by eq. 3.6, then we get:

$$\alpha \left(\frac{\Gamma(t)l(t)}{k(t)}\right)^{1-\alpha} = \gamma + \rho + \delta$$

$$\alpha \left(\frac{l(t)}{\widetilde{k}(t)}\right)^{1-\alpha} = \gamma + \rho + \delta$$

Thus,

$$l^s(a_r^*) = \left(\frac{\gamma + \rho + \delta}{\alpha}\right)^{\frac{1}{1-\alpha}} \widetilde{k}^*$$

Therefore, it is easy to see that:

$$\frac{\partial l^s(a_r^*)}{\partial n} = \left(\frac{\alpha}{\gamma + \rho + \delta}\right)^{\frac{1}{1-\alpha}} \frac{\partial \widetilde{k}^*}{\partial n} < 0$$

since $\frac{\partial \widetilde{k}^*}{\partial n} < 0$.

G. Proof of proposition 7.1

- $\frac{\partial a_r^*}{\partial \gamma} > 0$:

From the definition of the loci $\dot{a}_r= 0$ and $\dot{\widetilde{k}}= 0$, in the steady state is verified that:

$$\widetilde{k}\Big|_{\dot{a}_r=0} - \widetilde{k}\Big|_{\dot{\widetilde{k}}=0} = 0$$

By the Implicit Function Theorem:

$$\frac{\partial a_r^*}{\partial \gamma} = -\frac{\frac{\partial \widetilde{k}}{\partial a_r}\Big|_{\dot{a}_r=0} - \frac{\partial \widetilde{k}}{\partial a_r}\Big|_{\dot{\widetilde{k}}=0}}{\frac{\partial \widetilde{k}}{\partial \gamma}\Big|_{\dot{a}_r=0} - \frac{\partial \widetilde{k}}{\partial \gamma}\Big|_{\dot{\widetilde{k}}=0}} > 0$$

since $\left.\frac{\partial \widetilde{k}}{\partial a_r}\right|_{\dot{a}_r=0} < 0$, $\left.\frac{\partial \widetilde{k}}{\partial a_r}\right|_{\dot{\widetilde{k}}=0} > 0$, $\left.\frac{\partial \widetilde{k}}{\partial \gamma}\right|_{\dot{a}_r=0} > 0$, $\left.\frac{\partial \widetilde{k}}{\partial \gamma}\right|_{\dot{\widetilde{k}}=0} < 0$ (see the definition of the steady state in Appendix B for the calculations of the derivatives).

- $\frac{\partial \widetilde{k}^*}{\partial \gamma} < 0$:

  From Figure 1 it is easy to see that

$$\text{if} \quad \left| \left.\frac{\partial \widetilde{k}}{\partial \gamma}(\widetilde{k}^*)\right|_{\dot{\widetilde{k}}=0} \right| > \left.\frac{\partial \widetilde{k}}{\partial \gamma}(\widetilde{k}^*)\right|_{\dot{a}_r=0} \quad \text{then} \quad \frac{\partial \widetilde{k}^*}{\partial \gamma} < 0$$

We now proceed to prove that $\left| \left.\frac{\partial \widetilde{k}}{\partial \gamma}(\widetilde{k}^*)\right|_{\dot{\widetilde{k}}=0} \right| > \left.\frac{\partial \widetilde{k}}{\partial \gamma}(\widetilde{k}^*)\right|_{\dot{a}_r=0}$. Since $\frac{\partial l^s(a_r)}{\partial \gamma} < 0$ we can rewrite it as

$$\frac{1}{1-\alpha}\frac{\widetilde{k}}{(\delta+n+\gamma)} > \frac{\alpha}{1-\alpha}\widetilde{k}\left[\frac{1}{[(1-\alpha)(\delta+\rho+\gamma)+\alpha(\rho-n)]}\right]$$

simplifying:

$$(1-\alpha)(\delta+\rho+\gamma)+\alpha(\rho-n) > \alpha(\delta+n+\gamma)$$

and this is true since $\rho > n$ and $\alpha < \frac{1}{2}$.

- $\frac{\partial l^s(a_r^*)}{\partial \gamma} > 0$:

  From the definition of $l^s(.)$ in Appendix A it is easy to see that:

$$\frac{\partial l^s(a_r^*)}{\partial n} = \frac{\partial l^s(a_r^*)}{\partial a_r^*}\frac{\partial a_r^*}{\partial \gamma} = l(a_r^*)\mu(a_r^*)\frac{\partial a_r^*}{\partial \gamma} > 0$$

since $\frac{\partial a_r^*}{\partial \gamma} > 0$ as we proved in the first part of the proposition.

References

[1] Auerbach, A.J., L.J. Kotlikoff, R.P. Hagemann and G. Nicoletti (1989) "The economic dynamics of an ageing population: the case of four OECD countries", OECD Economic Studies, 12, 97130.

[2] Berkel, B., Börsch-Supan, A., Ludwing, A. and J. Winter (2004) "Sin die Probleme der Bevlkerungsalterung durch eine hhere Geburtenrate lsbar?", *Perspektiven der Wirtschaftspolitik*, 5(1), 71-90.

[3] Blanchet, D., Aubert, P., Blau, D. and C. Afsa (2005) "The labour market for older workers: some elements of a Franco-American comparison", OECD work in progress.

[4] Bloom D.E., Canning, D. and M. Moore (2004) "The Effect of Improvements in Health and Longevity on Optimal Retirement and Saving", NBER working paper n. 10919.

[5] Boadway, R., Marchand, M. and P. Pestieau (1990) "Optimal Path for Social Ssecurity in a Changing Environment", in Public Finance and Seteady Economic Growth, Krause-Junk G. ed, Foundation Journal of Public Finance, The Hague.

[6] Kohler, H.P., Billari, F.C and J.A. Ortega (2005) "Low Fertility in Europe: Causes, Implications and Policy Options", in The baby bust: Who will do the work? Who will pay the taxes?, Harris, F.R. ed, Rowman and Littlefeld Publishers, Inc.

[7] Boadway, R., Marchand, M. and P. Pestieau (1990) "Pay-as-you-go Social Ssecurity in a Changing Environment", *Journal of Population Economics*, 4, 257-280.

[8] Boldrin, M., Dolado, J.J., Jimeno, J.F. and F. Peracchi (2001) "The future of pensions in Europe", *Economic Policy*, 14(29), 287-320.

[9] Börsch-Supan, A. (2003) "Labor Market Effects of Population Aging", *Labour*, 17, 5-48.

[10] Conde-Ruiz, J.I. and V. Galasso (2003) "Early Retirement", *Review of Economic Dynamics*, 6, 12-36.

[11] Conde-Ruiz, J.I. and V. Galasso (2004) "The Macroeconomics of Early Retirement", *Journal of Public Economics*, 88, 1849-1869.

[12] Congress of United States. Congressional Budget Office (2004) "A CBO Study", 49pp.

[13] Crettez, B. and P. Le Maitre (2002) "Optimal age of retirement and population growth", *Journal of Population Economics*, 15, 737-755.

[14] Cutler, D.M., Poterba, J.M., Sheiner, L.M. and L.H. Summers (1990) "An ageing society: Opportunity or challenge?", *Brookings Papers on Economic Activity*, 1, 1-73.

[15] De Nardi, M.C., Imrohoroglu, S. and T. Sargent (1999) "Proyected US demographics and social security", *Review of Economic Dynamics*, 2, 575615.

[16] European Comission Survey (2004) "The Future of Pension Systems", *Eurobarometer*, 107pp.

[17] Fehr, H., Jokish, S., and L.J. Kotlikoff (2008) "Fertility, mortality and the developed world's demographic transition", *Journal of policy modeling*, 30, 455-473.

[18] Feldstein, M. and A. Samwick (1998) "The Transition Path in Privatizing Social Security", Privatizing Social Security, M. Feldstein ed, University of Chicago Press, Chicago.

[19] Feldstein, M. (1996). "The Missing Piece in Policy Analysis: Social Security Reform", *American Economic Review Papers and Proceedings*, 86(2), 1-14.

[20] Forman, J.B. and Y.P. Chen (2008) "Optimal Retirement Age", in New York Univerisity Review of Employee Benefits and Compensation, Alvin D. Lurie ed, New York.

[21] Fustos, K. (2010) "Do Family Policies Work? Evidence From Europe and the United States", Population Reference Bureau Communication.

[22] Grant, J., Hoorens, S., Sivadasan, S., Van Het Loo, M., Davanzo, J., Hale, L. and W. Butz (2006) "Trends in European fertility: should Europe try to increase its fertility rate or just manage the consequences?" *International journal of andrology*, 29, 17-24.

[23] Guest, R.S. and I.M. McDonald (2002) "Would a decrease in fertility be a thereat to living standrds in Australia?", *The Australian Economic Review*, 35, 29-44.

[24] Guest, R. (2006) "Population aging, capital mobility and optimal saving", *Journal of Policy Modeling*, 28, 89-102.

[25] Heijdra, B.J. and J.E. Ligthart (2006) "The macroeconomic dynamics of demographic shocks", *Macroeconomic Dynamics*, 10, 349-370.

[26] Hills, J. (2010) "An Anatomy of Economic Inequality in the UK - Report of the National Equality Panel", LSE STICERD research paper n. CASEREPORT60.

[27] Holzmann, R. (1997) "On Economic Benefits and Fiscal Requirements of Moving from Unfunded to Funded Pensions", *ECLAC Project Document*, 48.

[28] Kahn, J. (1988) "Social Security, Liquidity, and Early Retirement", *Journal of Public Economics*, 35, 97-117.

[29] Kohler, H.P., Billari, F.C and J.A. Ortega (2005) "Low Fertility in Europe: Causes, Implications and Policy Options", in The baby bust: Who will do the work? Who will pay the taxes?, Harris, F.R. ed, Rowman and Littlefeld Publishers, Inc.

[30] Kotlikoff, L. (1995) "Privatization of Social Security: How it Works and Why it Matter", NBER Working Paper No. 5330.

[31] Kotlikoff, L. (1996) "Privatizing Social Security at Home and Abroad", *American Economic Review Papers and Proceedings*, 86(2), 368-372.

[32] Kydland, F.E. (2004) "Quantitative Aggregate Theory", Prize Lecture.

[33] Lacomba, J.A. and F. Lagos (2006) "Population aging and legal retirement age", *Journal of Population Economics*, 19, 507-519.

[34] Luong, M. and Hérbert, B.P. (2009) "Age and Earnings", *Perspectives. Statistics Canada*, 75-001-X.

[35] Marchand, M., Michel, P., and P. Pestieau (1990) "Optimal Intergenerational Transfers in a Model with Fertility and Productivity Changes", CORE discussion paper n. 9059.

[36] Meijdam, L. and H.A.A. Verbon (1997) "Aging and Public Pensions in an Overlapping Generations Model", *Oxford Economic Papers*, 49, 29-42.

[37] Mitchell, O. and S. Zeldes (1996) "Social Security Privatization: A Structure for Analysis", *American Economic Review Papers and Proceedings*, 86(2), 363-367.

[38] OECD report (1998) "Work Force Ageing: Consequences and Policy Responses", Ageing working paper n. 4.1.

[39] OECD (1998) "Mataining prosperity in an ageing society", OECD, Paris.

[40] Prskawetz, A., Fent, T. and R. Guest (2008) "Workforce Aging and Labor Productivity. The role of supply and demand for labor in the G7", in Population Aging, Human Capital Accumulation and Productivity Growth, Prskawetz, Bloom and Lutz eds., New York: Population Council.

[41] Suzuki, T. (2007) "Lowest-Low Fertility and Governmental Actions in Japan", Center for Intergenerational Studies, Institute of Economic Research, Hitotsubashi University working paper n. 294.

[42] Velloso, H. (2005) "Social Security in the United States: Overview and Outlook", *ECLAC Project Document*, 89.

[43] Weller, C. (2005) "Raising the Retirement Age for Social Security: Implications for Low Wage, Minority, and Female Workers", *WP Center for American Progress*, 20pp.